Uniform Circular Motion

An object is in uniform circular motion if it is traveling at a constant speed on a circular path.

Example    An airplane is moving at a constant 90 m/s in a circle of radius 1000 m.

A motion that repeats over equal length intervals of time is called periodic. The distance the airplane must travel to complete one circle is $2\pi r = 2\pi(1000) = 6283$ m. Since the plane is flying at 90 m/s, it completes each trip around the circle in $6283/90 = 69.8$ s. This time of 69.8 s is the period of the motion. It is the amount of time for the plane to go through one complete cycle.

More generally, if an object is in uniform circular motion with speed $v$ on a circle of radius $r$, the period of the motion is

$$\text{period} = T = \frac{2\pi r}{v}$$

Uniform circular motion is called uniform because the speed is constant. The velocity, however, is always changing. A good way of visualizing velocity is an arrow starting at the object and pointing in the direction the object is moving.

As is seen in the picture above, the velocity is always changing in direction but is constant in magnitude.
An object experiencing an acceleration always changes its velocity. Whether or not the speed changes depends on the particular situation. As an example, suppose that you are driving your car north at a speed of 2.0 m/s (about 4.5 mi/h) and then turn west without slowing. Your speed in the northward direction changed from 2.0 m/s to 0 m/s while your speed in the westward direction changed from 0 m/s to 2.0 m/s. Therefore the car was experiencing two accelerations (northward and westward) while the overall speed stayed constant.

Uniform circular motion is another example of constant speed with non-zero acceleration. An acceleration that does not change the speed of an object is always perpendicular to the velocity and points in the direction the object is turning. For uniform circular motion, this means that acceleration points toward the center of the circle.

To find out how much acceleration is required to keep our object moving on its circular path, it is helpful to measure our angles in radians rather than degrees. Radians are an alternative to degrees that have the useful property that $s = r\theta$ for $s$, $r$, and $\theta$ as shown in the picture below.

Given an object in uniform circular motion, the distance $s$ that it moves along the circle is equal to its speed times the amount of time elapsed. In symbols this is $s = v\Delta t$. But since $s = r\theta$, we have $r\theta = v\Delta t$. Solving this equation gives us

$$\theta = \frac{v\Delta t}{r}$$
Next, if our object moves by an angle of \( \theta \), the velocity vector also changes by an angle of \( \theta \) as shown.

The change in velocity is shown below.

If \( \theta \) is small then \( \Delta v \) is approximately equal in length to the circular arc that joins the two velocity vectors. So, starting with the equation \( s = r\theta \), we notice that the radius is now \( v \) and thus

\[
\Delta v \approx s = v\theta
\]

But since \( \theta = v\Delta t/r \), we have

\[
\Delta v \approx v \cdot \frac{v\Delta t}{r}
\]

Dividing this equation by \( \Delta t \) gives us acceleration.

\[
a = \frac{\Delta v}{\Delta t} \approx \frac{1}{\Delta t} \frac{v^2 \Delta t}{r} = \frac{v^2}{r}
\]

Even though we only have an approximation, it turns out that this expression gives us the exact instantaneous acceleration. In summary

If an object is in uniform circular motion with speed \( v \), it has an acceleration with direction toward the center of the circle and magnitude given by

\[
a = \frac{v^2}{r}
\]

If an object has an acceleration \( a = v^2/r \), it follows that the net force acting on the object is
\[ F_c = \frac{mv^2}{r} \]

where the subscript \( c \) is used because this is known as the centripetal force.

**Example**  It is impossible for a car to turn at right angles. So suppose that as a car is turning, it is moving along a circular arc of radius \( r \). If the speed of the car is 3.0 m/s and the coefficient of static friction between the tires and the road is 0.45, what is the smallest radius the car may turn without sliding?

The centripetal force required for the turn is

\[ F_c = \frac{mv^2}{r} = \frac{m3.0^2}{r} \]

The maximum frictional force is

\[ f = \mu_s F_N = 0.45mg \]

When the maximum frictional force is equal to the centripetal force we have

\[ \frac{m3.0^2}{r} = 0.45mg \]

Solving this equation, we get \( r = \frac{3.0^2}{0.45g} = \frac{2.0}{m} \).

**Example**  A satellite has an orbital period of 15 hours. How high is the satellite above the surface of the earth?

The speed of the satellite is \( v = \frac{2\pi r}{T} = \frac{2\pi r}{(15 \cdot 3600)} = 0.000116r \) where \( r \) is the distance from the satellite to the center of the earth.

The centripetal force to keep the satellite in orbit must be supplied by gravity. Thus

\[ \frac{mv^2}{r} = \frac{GMm}{r^2} \]

where \( m \) is the mass of the satellite and \( M \) is the mass of the earth. Solving this equation we get

\[ r = \frac{GM}{v^2} = \frac{GM}{(0.000116r)^2} \]

Then
From the cover of the textbook we see that $M = 5.98 \times 10^{24}$ kg and $G = 6.673 \times 10^{-11}$ N $\cdot$ m$^2$/kg$^2$. So

$$r^3 = \frac{GM}{0.000116^2}$$

$$r = \sqrt[3]{\frac{GM}{0.000116^2}} = \sqrt[3]{\frac{(6.673 \times 10^{-11})(5.98 \times 10^{24})}{0.000116^2}} = 3.1 \times 10^7 \text{ m}$$

Since the radius of the earth is $6.38 \times 10^6$ m, the height of the satellite is $3.1 \times 10^7 - 6.38 \times 10^6 = 2.5 \times 10^7 \text{ m}$.

**Example**  The earth rotates once per day about an axis passing through the north and south poles, an axis that is perpendicular to the plane of the equator. Assuming the earth is a sphere with a radius of $6.38 \times 10^6$ m, determine the speed and centripetal acceleration of a person situated (a) at the equator and (b) at a latitude of $30.0^\circ$ north of the equator.

Each day is equal to 86400 s. For a person at the equator, $r = 6.38 \times 10^6$ m. The speed of the person is

$$v = \frac{2\pi r}{T} = \frac{2\pi (6.38 \times 10^6)}{86400} = 464.0 \text{ m/s}$$

The centripetal acceleration is

$$a = \frac{v^2}{r} = \frac{464.0^2}{6.38 \times 10^6} = 0.0337 \text{ m/s}^2$$

For a person at $30.0^\circ$, $r = (6.38 \times 10^6) \cos 30.0^\circ = 5.525 \times 10^6$ m. The speed of the person is

$$v = \frac{2\pi r}{T} = \frac{2\pi (5.525 \times 10^6)}{86400} = 401.8 \text{ m/s}$$

The centripetal acceleration is

$$a = \frac{v^2}{r} = \frac{401.8^2}{5.525 \times 10^6} = 0.0292 \text{ m/s}^2$$