Trigonometry

If \( \theta \) is one of the (non 90°) angles of a right triangle, the three sides may be labeled as hypotenuse, opposite, and adjacent:

The trigonometric functions sine, cosine, and tangent are defined as ratios of the sides of the right triangle:

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}
\]

Example

Suppose that \( \theta \) is the angle shown in the right triangle below.

Then the trigonometric functions have the following values.

\[
\sin \theta = \frac{3}{5} \quad \cos \theta = \frac{4}{5} \quad \tan \theta = \frac{3}{4}
\]

By using the inverse trigonometric keys on a calculator we can find the value of \( \theta \):

\[
\theta = \sin^{-1} \frac{3}{5} = \cos^{-1} \frac{4}{5} = \tan^{-1} \frac{3}{4} = 36.9^\circ
\]

Example

What is the distance between A and C?

\[
\text{A} \quad 10 \text{ km} \quad \text{B}
\]

\[
\text{A} \to \text{C} \to \text{B}
\]
Solution:

We can cut the triangle in half to produce a right triangle:

![Diagram of a right triangle]

Since we know the adjacent side is 5 km, we will use the cosine function to find $h$:

\[
\cos 25^\circ = \frac{5}{h}
\]

\[
h = \frac{5}{\cos 25^\circ} = 5.52 \text{ km}
\]

**Pythagorean Theorem**  The sides of a right triangle satisfy the equation $c^2 = a^2 + b^2$ where $c$ is the hypotenuse.

![Diagram of a right triangle]

**Law of Cosines**  Given any triangle, the sides satisfy the equation $c^2 = a^2 + b^2 - 2ab \cos \theta$ where $c$ is the side opposite the angle $\theta$.

![Diagram of a general triangle]

**Example**  Find $c$.  

![Diagram of a triangle with sides 4 m, 3 m, and angle 70°]
Solution:

\[ c^2 = 4^2 + 3^2 - 2 \cdot 4 \cdot 3 \cos 70^\circ = 16.7915 \]
\[ c = \sqrt{4^2 + 3^2 - 2 \cdot 4 \cdot 3 \cos 70^\circ} = 4.10 \text{ m} \]

Vectors

A vector is a combination of a number and a direction. For example, if you walk 15 m northeast, this describes a vector. The number (15 m) is called the magnitude of the vector. A graphical representation of the vector is an arrow of length 15 m pointing in the direction northeast.

In contrast to a vector, a number without direction is called a scalar. Measurable quantities are often easily classified as either vectors or scalars. For example, force, velocity, acceleration, and momentum are vectors since each has a natural direction, but quantities such as time, temperature, and mass usually have no direction and thus are scalars.

Examples

Vectors:

1. Displacement: The change in an object’s position. The vector arrow connects the initial and final positions of the object.

2. Velocity: Change in position divided by change in time. The arrow begins at the object’s current position and points in the direction the object is moving. The length of the arrow is the speed of the object.

3. Acceleration: Change in velocity divided by change in time. Acceleration is a measure of how quickly velocity is changing.
4. Force: A measure of ability to change an object’s velocity. We will discuss force in greater detail later.

**Scalars:**

1. Time: Both moments in time and intervals of time.
2. Mass: A property of an object that quantifies its resistance to changes in velocity.

**Vector Algebra**

**Notation:** Vectors are represented by boldface letters (\( \mathbf{v} \)) or by letters with an overline arrow (\( \overline{v} \)). The magnitude of the vector \( \mathbf{v} \) is written as \( |\mathbf{v}| \) or more simply as non-boldface \( v \). We consider two vectors to be equal if they have the same magnitude and direction. Thus, the two vectors shown below are equal.

When adding two vectors \( \mathbf{v} \) and \( \mathbf{w} \), we first draw the vectors so that the ending point of one vector is the beginning point of the other vector:

The sum \( \mathbf{v} + \mathbf{w} \) is a vector that starts at the same place \( \mathbf{v} \) starts and ends at the same place \( \mathbf{w} \) ends:
In what we just did, the vectors $\mathbf{v}$ and $\mathbf{w}$ are called components of $\mathbf{v} + \mathbf{w}$ because the vector $\mathbf{v} + \mathbf{w}$ is the sum of $\mathbf{v}$ and $\mathbf{w}$. A standard procedure for working with a vector $\mathbf{u}$ is to write it as the sum of two components $\mathbf{u} = \mathbf{u}_x + \mathbf{u}_y$ where $\mathbf{u}_x$ is a vector parallel to the $x$ axis and $\mathbf{u}_y$ is a vector parallel to the $y$ axis:

Once this is done we can write $\mathbf{u} = (u_x, u_y)$ where $u_x$ is the magnitude of $\mathbf{u}_x$ and $u_y$ is the magnitude of $\mathbf{u}_y$.

**Example**

Let $\mathbf{v}$ be a vector with magnitude 3 and direction as shown below.

Next, draw in the $x$ and $y$ components of $\mathbf{v}$:

Since $v_y$ is the opposite side, it is given by the equation
\[ v_y = 3 \sin 33.7^\circ = 1.66 \]

and \( v_x \) is given by

\[ v_x = 3 \cos 33.7^\circ = 2.50 \]

Thus, by using the trigonometric functions, we have found the components of a vector. We can now write \( \mathbf{v} = (2.50, 1.66) \).

**Addition**

If we want to add two vectors \( \mathbf{v} \) and \( \mathbf{u} \), we first find the components: \( \mathbf{v} = (v_x, v_y) \) and \( \mathbf{u} = (u_x, u_y) \). The sum of the vectors is given by \( \mathbf{v} + \mathbf{u} = (v_x + u_x, v_y + u_y) \).

**Example**

Add the two vectors shown below given that \( v = 2 \) and \( u = 1.5 \).

![Vectors v and u](image)

**Solution:**

The components of \( \mathbf{v} \):

\[ v_x = 2 \cos 20^\circ = 1.879 \]
\[ v_y = 2 \sin 20^\circ = .684 \]

The components of \( \mathbf{u} \):

\[ u_x = 1.5 \cos 55^\circ = .860 \]
\[ u_y = 1.5 \sin 55^\circ = 1.229 \]

Let \( \mathbf{R} \) be the name of the sum: \( \mathbf{R} = \mathbf{v} + \mathbf{u} \). Then

\[ \mathbf{R} = (v_x + u_x, v_y + u_y) = (2.739, 1.913) \]
The magnitude of $\mathbf{R}$ is given by the Pythagorean theorem:

$$|\mathbf{R}| = \sqrt{2.739^2 + 1.913^2} = 3.341$$

The angle of $\mathbf{R}$ is given by the equation $\tan \theta = 1.913/2.739$.

$$\theta = \tan^{-1} \frac{1.913}{2.739} = 34.9^\circ$$